Precautionary Saving at the Top: On the Concavity of Consumption in the Permanent Component of Earnings

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Abstract

Changes in the distribution of permanent income, which is a proxy for human capital, have become a key explainer of wealth accumulation at the top, but we know little about the theoretical relation between permanent income and consumption in economic models. I show that, in a standard model, consumption is concave in the permanent component of income: the marginal propensity to consume out of permanent income decreases with permanent income. The concavity arises because an increase in permanent income corresponds to an increase in uncertain future income. Thus, permanent income raises consumption because it raises total resources but its effect is dampened by the fact that it also raises the need for precautionary saving. Under standard preference assumptions, the increase in precautionary saving is higher at a higher initial level of permanent income, that is, for those who are initially exposed to more income risk. This contrasts with the concavity in accumulated wealth, which arises because precautionary saving decreases with wealth, but less so at a higher level of wealth. This reshapes the view on who engages in precautionary saving: there can be precautionary saving at the top of the income distribution. It also highlights what components of a model are important for fully capturing these concavities and why current frameworks may not capture it.

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1 Introduction

The distribution of the permanent component of people's income has changed substantially over the past decades, with potentially major macroeconomic consequences. The permanent component of income is the component that captures the return to permanent skills.[1] It typically multiplies all realizations of an individual's income, that is, it multiplies human capital. For many people, at a given point in time, most of their total expected resources comes from their expected flow of future income—their human capital. A number of studies have documented a substantial increase in the inequality with which the permanent component of labor income is distributed across workers (Kopczuk, Saez, and Song 2010; DeBacker, Heim, Panousi, Ramnath, and Vidangos 2013; Guvenen, Kaplan, Song, and Weidner 2022). This increase in permanent income inequality has been shown to have the potential to drive profound economic changes. In numerical simulations, paired with non-homothetic preferences, this increase in permanent income inequality jumps-starts a cycle of wealth accumulation. This initial trigger can then account for trends such as the secular increase in wealth inequality, the decrease in interest rate, and the rise in aggregate debt (Straub 2019; Mian, Straub, and Sufi 2021).

Despite this rising importance of the distribution of permanent income in macroeconomic models, we know little about the effect of permanent income on consumers' decisions. Models that use the change in permanent income distribution as a starting point often treat income in a stylized way, without risk. As a result, an increase in income is equivalent to an increase in accumulated wealth up to borrowing or pledgability constraints.

In this paper, I argue that an increase in permanent income can go hand-in-hand with an increase in saving, in particular at the top of the permanent income distribution, and so only through precautionary behavior without departing from homothetic preferences and without borrowing constraints. I show this in three steps. First, I prove that consumption is concave in permanent income: the marginal propensity to consume out of permanent income decreases with permanent income. This echoes the result of Carroll and Kimball (1996) that consumption is concave in wealth. I extend both concavity results to non-homothetic preferences but in that case the effects interact with the patience and individual asset return of the consumer. Specifically, the concavity may only hold for patient enough consumers or those with high enough asset returns. Second, I show that consumption is

[1] This is distinct from the permanent income defined in Friedman (1957), which is the amount of resources that a consumer facing no uncertainty optimally chooses to consume.

concave in permanent income but for a reason that is the opposite of the one behind the concavity in wealth. For the most common preferences used in the literature, consumption is concave in wealth because wealth reduces precautionary saving but less and less so as wealth increases. In contrast, for the same preferences, an increase in permanent income raises precautionary saving, and more so at higher levels of permanent income. This is because permanent income scales up risky human capital so it increases the exposure to future shocks. There can be precautionary saving at the top. Third, I note that the existing literature does not give a chance to this channel. This is because the models either eliminate or do not fully account for the true level of income risk that people face, or do not give consumers access to a risk-free asset they could save in for precautionary reasons.

I consider a life-cycle model in which people consume in order to maximize their lifetime expected utility. They face uninsurable income shocks and have access to one risk-free asset. The return on this risk-free asset can be individual-specific.

In this model, consumption is concave in permanent income for a general class of utility functions This general class includes Hyperbolic Absolute Risk Aversion (HARA), to which the most standard preferences such as quadratic, exponential or isoelastic functions belong, but also non-HARA utility functions that can capture non-homotheticicities. For non-HARA functions, the value of the interest rate and discount factor matter. In particular, for a utility function that can encapsulate non-homotheticies, the concavity is unambiguous only for consumers with high levels of patience or high individual returns. I extend the proof that consumption is concave in wealth, established for HARA functions in the seminal paper of (Carroll and Kimball (1996)), to that broader class of functions. The same result that, for non-HARA, the value of the interest rate and discount factor matter holds in the case of the concavity in wealth.

I then examine the reasons behind the concavity in permanent income and wealth. We know that the concavities must be driven by precautionary behavior. Indeed, absent uncertainty, consumption is linear in wealth and permanent income. However, to the best of my knowledge, there is no correct proof on how precautionary saving evolves with wealth.[2] I show that, under some conditions on preferences, the reason for the concavity in wealth

[2]The Lemma 3 in Carroll, Holm, and Kimball (2021) states that consumption function in the presence of risk is a counterclockwise concavification around ∞ of the consumption function in the absence of risk. This would mean that precautionary saving decreases with wealth. However, to prove this concavification, they only state that precautionary saving is positive and consumption is concave in wealth. Those two things are also true in the case of permanent income. Yet the presence of risk does not concavify the relation between consumption and permanent income around ∞.

is that an increase in wealth reduces precautionary saving, but less so at higher levels of wealth. In contrast, the reason for the concavity in permanent income is that an increase in permanent income raises precautionary saving, and more so at a higher level of permanent income. This reshapes the view on who engages in precautionary saving: at given level of wealth, precautionary saving increases with permanent income. This leads to wealth accumulation among those with higher income, although this wealth accumulation itself slows as people with higher permanent income's wealth increases.

I use these results to discuss existing empirical analyses and models. These results mean that, to give the standard model a chance to generate a realistic degree of concavity in permanent income, it is crucial to adequately model the income risk that people face and also to let consumers access a safe asset they can use for precautionary saving. Yet, this combination rarely exists. First, the models preoccupied with modeling how the increase in inequality can trigger wealth accumulation and the related macro-adjustments typically neglect risk. Straub (2019) includes only normal shocks and Mian, Straub, and Sufi (2021) do not include any income risk. Since in the absence of risk an increase in permanent income is the same as an increase in wealth, it then reduces precautionary saving: it cannot trigger any meaningful wealth accumulation and non-homothetic preferences are the only way to generate saving at the top.

Second, there is a large literature that has tried to understand the mapping between wealth and income inequalities. These show that entrepreneurship as well as heterogeneous returns can generate high income inequalities, translating into large wealth inequalities (Benhabib, Bisin, and Zhu 2011; Benhabib, Bisin, and Luo 2017; Guvenen, Kambourov, Kuruscu, Ocampo-Diaz, and Chen 2023; Gaillard, Hellwing, Wangner, and Werquin 2023). This literature initially did examine the levels of saving and consumption at the top implied by their models. In their pioneer paper, Gaillard, Hellwing, Wangner, and Werquin (2023) also considers this dimension. They document that, although heterogeneous returns can match the Pareto tails of capital income and wealth that they measure empirically, such a model cannot match the Pareto tail of consumption: non-homothetic preferences are required to match it. I note that their model has only one asset, which is risky and even increasingly risky as capital income increases. Thus, in this model, consumers with high capital income do not have access to any risk-free asset to make precautionary saving. This does not make it possible for saving to increase via the mechanism I describe. Further, I show that, in a setting with only one risky asset available, it is even possible to obtain negative precautionary saving: saving less in the risky asset than one would have absent uncertainty can be a strategy to mitigate risk.

2 Model

I consider a standard consumption model with a transitory-permanent earnings process and only one risk-free asset. A consumer i is finite-lived, with T the length of their life. The consumer chooses consumption expenditures at period t, denoted c_t^i, to maximize lifetime expected utility subject to a number of constraints

$$V_t^i(a_t^i, e^{p_t^i}, e^{\varepsilon_t^i}) = \max_c \; u(c) + \beta E_t \left[V_{t+1}^i(a_{t+1}^i, e^{p_{t+1}^i}, e^{\varepsilon_{t+1}^i}) \right] \tag{2.1}$$

$$with \text{ Utility conditions: } u'(.) > 0, u''(.) < 0, \text{ and } u'''(.) > 0 \tag{2.2}$$

$$\text{Budget constraint: } a_{t+1}^i = (1+r)a_t^i + y_t^i - c, \tag{2.3}$$

$$\text{Earnings: } y_t^i = e^{\Gamma_t} e^{p_t^i} e^{\varepsilon_t^i}, \;\; var_t^i(\varepsilon_t^i) > 0 \tag{2.4}$$

$$\text{Permanent component: } e^{p_{t+1}^i} = e^{p_t^i} e^{\eta_{t+1}^i}, \tag{2.5}$$

$$\text{Terminal wealth: } a_{T+1}^i \geq 0. \tag{2.6}$$

Utility is time-separable and at each period depends only on contemporaneous consumption. The period utility function $u(.)$ is such that marginal utility is positive, decreasing, and convex in consumption. The discount factor β captures how much consumers discount utility between two consecutive periods.

The budget constraint (2.3) states that, to store their wealth from one period to the next the consumer only has access to one risk-free liquid asset. The term a_t^i denotes the level of this asset at the beginning of period t—or at the end of $t-1$. The risk-free return rate is r. This rate r is such that $\beta(1+r) \leq 1$.

The income specification, described with (2.4) and (2.5), means that income is a transitory-permanent process with a time trend: income is the product of a deterministic time trend e^{Γ_t}, a permanent component $e^{p_t^i}$ that evolves as a multiplicative random walk and of a transitory innovation $e^{\varepsilon_t^i}$. Because the permanent component $e^{p_t^i}$ multiplies the value of the permanent component at the next period, it multiplies each realization of earnings until the rest of the consumer's lifetime: at $t+s$, earnings are $y_{t+s}^i = e^{\Gamma_{t+s}} e^{p_t^i} e^{\eta_{t+1}^i + \ldots + \eta_{t+s}^i} e^{\varepsilon_t^i}$. It thus plays the role of a scaling factor. Note that this specification encompasses an even simpler specification in which the permanent component is just a multiplicative fixed effect $e^{p_t^i} = e^{p^i}$. This is for instance the specification in Straub (2019). I do not impose any particular distribution for the innovations. They could for instance be drawn from mixture of normal distributions—as in Guvenen, Karahan, Ozkan, and Song (2021) who show that such distributions help match important features of observed earnings distributions. For the precautionary motive to be strictly positive, I impose that each individual faces a strictly positive amount of transitory earnings uncertainty: $var_t^i(\varepsilon_t^i) > 0$.

The terminal condition on wealth (2.6) states that the consumer cannot die with a strictly positive level of debt: assets at the end of the last period T—and the beginning of $T+1$—have to be non-negative. The combination of this condition with the period budget constraints and positive spending constraints generates a natural borrowing constraint that prevents people from holding a level of debt superior to what they could ever repay. This constraint never binds because marginal utility approaches infinity as consumption approaches zero: consumers would never put themselves in the situation of possibly consuming zero in the future.

First order condition. The first order condition of the problem is

$$u'(c_t^i) = E_t^i[u'(c_{t+1}^i)]\underbrace{\beta(1+r)}_{R} \tag{2.7}$$

E_t^i denotes the expectation operator of consumer i at period t. The condition states that, to maximize their total expected utility, consumers seek to equalize their current and expected future marginal utility of consumption, weighted by a deterministic term R that captures the effect of people's discount factor β relative to that of the interest rate $(1+r)$.

Precautionary saving. Precautionary saving is commonly defined as the 'additional saving that results from the knowledge that the future is uncertain' (Carroll and Kimball (2006)). This characterization comes with some ambiguity, however, because there are many ways to define the counterfactual world in which the future is not uncertain.[3]

I use the definition that is the most common in recent studies—at least in those relying

[3]For example, Drèze and Modigliani (1972) study the change in consumption observed when removing uncertainty in a way that keep expected lifetime utility the same, while Kimball (1990) studies the shift in assets required for households facing some uncertainty to enjoy the same level of consumption as they would in the absence of uncertainty. Even when one decides to simply set exogenous variables equal to their expected value with probability one, one has to make a choice: some exogenous and uncertain variables are non-linear functions of other exogenous and uncertain variables so they cannot all be equal to their expected value at the same time.

on an exogenous income process—: the certain counterfactual is the one in which future income equals its expected value with probability one, everything else being equal. I denote with an index PF_t, for perfect foresight, the value that this variable would take if, from period t on, future income was equal to expected value:[4]

$$c_t^{i,PF_t} = c_t{}^i_{|y^i_{t+s}=E_t[y^i_{t+s}]\ \forall s>0}$$

Precautionary saving at t, denoted PS_t, is then the difference between the consumption that would take place under perfect foresight at t and actual consumption at t:

$$PS_t^i = c_t^{i,PF_t} - c_t^i \tag{2.8}$$

Because $var_t^i(\varepsilon_t^i) > 0$, future consumption is strictly uncertain. At the last period, there is no uncertainty so $c_T = c_T^{PF_T}$. I assume that $c_{t+1} \leq c_{t+1}^{PF_{t+1}}$ and show that the Euler equation then implies $c_t < c_{t<}^{PF_t}$. First, I use Jensen's inequality, then I plug in the previous inequality, and I note that $E_t^i[c_{t+1}^{i,PF_{t+1}}] = c_{t+1}^{i,PF_t}$, and

$$u'(c_t^i) = E_t^i[u'(c_{t+1}^i)]R > u'(E_t^i[c_{t+1}^i])R \geq u'(E_t^i[c_{t+1}^{i,PF_{t+1}}])R = u'(c_{t+1}^{i,PF_t})R \tag{2.9}$$

$$c_t^i < (u')^{-1}\big(u'(c_{t+1}^{i,PF_t})R\big) = c_t^{i,PF_t}. \tag{2.10}$$

Thus, precautionary saving is strictly positive at any $t < T$

$$PS_t^i = c_t^{i,PF_t} - c_t^i > 0. \tag{2.11}$$

3 Proving the concavity in permanent income and extending the concavity in wealth

3.1 Concavity in permanent income

Lemma 1. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when (i) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, and (ii) the ratio of temperance over prudence $(-u''''(.))(-u''(.))/u'''(.)^2$ is constant or

[4] The time appears in the index because it can make a difference: assuming that income is equal to expected value from period t on is not the same as assuming income is equal to expected value from period $t+s$ on.

decreasing, then

$$E_t[(-u''(c_{t+1}))^2/u'''(c_{t+1})]R \leq (-u''(c_t))^2/u'''(c_t).$$

Proof of Lemma 1. The proof of Lemma 1 is in the Online Appendix (A.1).

Lemma 2. In the model described above by (2.1)-(2.6), at any period $t \leq T$

$$\frac{\partial^2 c_{t+1}}{\partial a_{t+1} \partial e^{p_{t+1}}} \leq \sqrt{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} \frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}.$$

Proof of Lemma 2. The proof of Lemma 2 is in the Online Appendix (A.2).

Proposition 1. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when the ratios of temperance over prudence $(-u''''(.))(-u''(.))/u'''(.)^2$ and prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ are constant or decreasing, consumption is concave in permanent earnings

$$\frac{\partial^2 c_t}{\partial (e^{p_t})^2} \leq 0.$$

In addition, when the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is not always equal to one, the concavity is strict

$$\frac{\partial^2 c_t}{\partial (e^{p_t})^2} < 0.$$

Proof of Proposition 1. The proof is by backward induction. At the last period, consumers consume all of their remaining resources so $c_T = (1+r)a_T + e^{\Gamma_T} e^{\varepsilon_T} e^{p_T}$. Consumption is then linear in e^{p_T}: $(\partial^2 c_T)/(\partial (e^{p_T})^2) = 0$. Proposition 1 is true at $t = T$. I then assume that Proposition 1 is true at $t+1$ and prove that it must be true at t. I derive both sides of the Euler equation (2.7) with respect to a change in e^{p_t}

$$\frac{\partial c_t}{\partial e^{p_t}}(-u''(c_t)) = E_t[(\frac{\partial a_{t+1}}{\partial e^{p_t}} \frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}} \frac{\partial c_{t+1}}{\partial e^{p_{t+1}}})(-u''(c_{t+1}))]R \tag{3.1}$$

I derive each side a second time with respect to e^{p_t}

$$\frac{\partial^2 c_t}{\partial (e^{p_t})^2}(-u''(c_t)) - \left(\frac{\partial c_t}{\partial e^{p_t}}\right)^2 u'''(c_t) = E_t\Bigg[\Bigg(\frac{\partial^2 a_{t+1}}{\partial (e^{p_t})^2}\frac{\partial c_{t+1}}{\partial a_{t+1}} + (\frac{\partial a_{t+1}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} \\ + \frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + \frac{\partial^2 e^{p_{t+1}}}{\partial (e^{p_t})^2}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} + \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} \\ + (\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\Bigg)(-u''(c_{t+1}))\Bigg]R - E_t\Bigg[(\frac{\partial c_{t+1}}{\partial e^{p_t}})^2 u'''(c_{t+1})\Bigg]R. \tag{3.2}$$

I substitute $\frac{\partial^2 a_{t+1}}{\partial (e^{p_t})^2} = -\frac{\partial^2 c_t}{\partial (e^{p_t})^2}$ (from the differentiated budget constraint at t), $\frac{\partial c_t}{\partial e^{p_t}} = E_t[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}]$ (from the differentiated Euler equation at t) and $\frac{\partial^2 e^{p_{t+1}}}{\partial (e^{p_t})^2} = 0$ (from the twice differentiated evolution of permanent earnings over time). I note that the last term is equivalently written as multiplied by 1 and substitute using the inequality $1 \geq E_t[\frac{(-u''(c_{t+1}))^2/u'''(c_{t+1})}{(-u''(c_t))^2/u'(c_t)}]R$ (from Lemma 1). I divide all sides by $(-u''(c_t))$ and rearrange

$$\frac{\partial^2 c_t}{\partial (e^{p_t})^2}\left(1 + E_t\left[\underbrace{\frac{\partial c_{t+1}}{\partial a_{t+1}}}_{>0}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) \tag{3.3}$$
$$\leq E_t\Bigg[\underbrace{-\Bigg((\frac{\partial a_{t+1}}{\partial e^{p_t}})^2(-\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}) - 2\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + (\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2(-\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2})\Bigg)}_{\leq -\left(\frac{\partial a_{t+1}}{\partial e^{p_t}}\sqrt{\left(-\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\right)} - \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\sqrt{\left(-\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\right)}\right)^2 \leq 0 \text{ using Lemma 2}}\frac{-u''(c_{t+1})}{-u''(c_t)}\Bigg]R$$
$$-\frac{u'''(c_t)}{-u''(c_t)}\Bigg(\underbrace{E_t\left[(\frac{\partial c_{t+1}}{\partial e^{p_t}})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right]E_t\left[\frac{(-u''(c_{t+1}))^2/u'''(c_{t+1})}{(-u''(c_t))^2/u'(c_t)}\right]R^2 - E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2 R^2}_{\geq 0 \text{ from Cauchy-Schwarz}}\Bigg) \leq 0.$$

Indeed, Lemma 2 states that, at any $t < T$, $\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} \leq \sqrt{-\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}}\sqrt{-\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}$. This means that the expression on the second line factorizes as the opposite of a square, which is always negative. Finally, using the more compact notations $P_{t+1} = \frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{\sqrt{u'''(c_{t+1})}}{\sqrt{u'''(c_t)}}$ and $U_{t+1} = \frac{(-u''(c_{t+1})/\sqrt{u'''(c_{t+1})}}{(-u''(c_t)/\sqrt{u'''(c_t)}}$, the last term rewrites $E_t[P_{t+1}^2]E_t[U_{t+1}^2]R^2 - E_t[P_{t+1}U_{t+1}]^2R^2$. The Cauchy-Schwarz inequality implies that this term is positive. Thus Proposition 1 is true at any period $t \leq T$.

Strict concavity. To extend to the case of strict concavity, one needs to exclude the case in which $u'''(.)u'(.)/(-u''(.))^2$ is always equal to one. This coincides to the case when utility displays Constant Absolute Risk Aversion (CARA). Indeed, when $u'''(.)u'(.)/(-u''(.))^2$ is constantly equal to one, the terms P_{t+1} and U_{t+1} are linearly dependent and such that $E_t[P_{t+1}^2]E_t[U_{t+1}^2]R^2 = E_t[P_{t+1}U_{t+1}]^2R^2$. Also, with CARA utility, Lemma 1 and Lemma 2 yield equalities as well—Lemma 2 relies on a similar Cauchy-Schwarz inequality. As a result, all inequalities are equalities, and consumption is linear in permanent income. Note that, under the assumption that $u'''(.) > 0$, $-u''(.) > 0$, and $u'(.) > 0$, it's not possible that $u'''(.)u'(.)/(-u''(.))^2$ be constant and equal to zero (quadratic utility). That's why, although this also yields a linear function, it is not part of the cases that need to be excluded to obtain a strict concavity.

Discussion. Proposition 2 establishes concavity in permanent income: in the same way that people consume less of an increase in wealth at higher levels of wealth, they consume less of an increase in permanent income at a higher level of permanent income. This proves for permanent income what Carroll and Kimball (1996) proves for wealth, and proves it for a more general class of utility functions.

3.2 Concavity in wealth

Proposition 2. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when (i) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, and (ii) the ratio of temperance over prudence $(-u''''(.))(-u''(.))/u'''(.)^2$ is constant or decreasing, then consumption is concave in wealth

$$\frac{\partial^2 c_t}{\partial a_t^2} \leq 0.$$

In addition, when the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is not always equal to one, the concavity is strict

$$\frac{\partial^2 c_t}{\partial a_t^2} < 0.$$

Proof of Proposition 2. The proof is by backward induction. At the last period, consumers consume all of their remaining resources $c_T = (1+r)a_T + e^{\Gamma_T}e^{\varepsilon_T}e^{p_T}$. Consumption is linear in a_T and e^{p_T}, so $(\partial^2 c_T)/(\partial a_T^2) = 0$. Proposition 2 is true at $t = T$. I assume that Proposition 2 is true at $t+1$ and prove that it must be true at t. I derive both sides of the Euler equation (2.7) with respect to a change in a_t

$$\frac{\partial c_t}{\partial a_t}(-u''(c_t)) = E_t\left[\frac{\partial a_{t+1}}{\partial a_t}\frac{\partial c_{t+1}}{\partial a_{t+1}}(-u''(c_{t+1}))\right]R \tag{3.4}$$

I derive each side a second time with respect to a_t

$$\begin{aligned}&\frac{\partial^2 c_t}{\partial a_t^2}(-u''(c_t)) - \left(\frac{\partial c_t}{\partial a_t}\right)^2 u'''(c_t) \\ &\quad = E_t\left[\left(\frac{\partial^2 a_{t+1}}{\partial a_t^2}\frac{\partial c_{t+1}}{\partial a_{t+1}} + (\frac{\partial a_{t+1}}{\partial a_t})^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\right)(-u''(c_{t+1}))\right]R - E_t\left[(\frac{\partial c_{t+1}}{\partial a_t})^2 u'''(c_{t+1})\right]R.\end{aligned} \tag{3.5}$$

I substitute $\frac{\partial^2 a_{t+1}}{\partial a_t^2} = -\frac{\partial^2 c_t}{\partial a_t^2}$ (from the budget constraint at t differentiated with respect to a_t), $\frac{\partial c_t}{\partial a_t} = E_t[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}]$ (from the Euler equation at t differentiated with respect to a_t). Without loss of generality, the last term is multiplied by 1, and I substitute using $1 \geq E_t[\frac{(-u''(c_{t+1}))^2/u'''(c_{t+1})}{(-u''(c_t))^2/u'(c_t)}]R$ (from Lemma 1). I divide all sides by $(-u''(c_t))$ and rearrange

$$\begin{aligned}&\frac{\partial^2 c_t}{\partial a_t^2}\left(1 + E_t\left[\underbrace{\frac{\partial c_{t+1}}{\partial a_{t+1}}}_{>0}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R\right) \leq E_t\left[(\frac{\partial a_{t+1}}{\partial a_t})^2\underbrace{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}}_{\leq 0}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R \\ &- \frac{u'''(c_t)}{-u''(c_t)}\left(\underbrace{E_t\left[(\frac{\partial c_{t+1}}{\partial a_t})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right]E_t\left[\frac{(-u''(c_{t+1}))^2/u'''(c_{t+1})}{(-u''(c_t))^2/u'(c_t)}\right]R^2 - E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2 R^2}_{\geq 0 \text{ from Cauchy-Schwarz}}\right) \leq 0.\end{aligned} \tag{3.6}$$

The second order effect of wealth on consumption is negative. Indeed, the term that multiplies it is positive because the effect of an increase in wealth on consumption is strictly positive at any period.[5] The first term on the right-hand side is negative because by assumption Proposition 2 holds at $t1$. The second term on the right-hand side is negative as well: using the more compact notations $A_{t+1} = \frac{\partial c_{t+1}}{\partial a_t}\frac{\sqrt{u'''(c_{t+1})}}{\sqrt{u'''(c_t)}}$ and $U_{t+1} = \frac{(-u''(c_{t+1})/\sqrt{u'''(c_{t+1})}}{(-u''(c_t)/\sqrt{u'''(c_t)}}$, the last term rewrites $E_t[A_{t+1}^2]E_t[U_{t+1}^2]R^2 - E_t[A_{t+1}U_{t+1}]^2R^2$. The Cauchy-Schwarz inequality implies that this term is positive so its opposite is negative. Thus Proposition 1 is true at t when it is true at $t+1$. This means that it is true at any period $t \leq T$.

[5] This can be derived from the Euler equation at $t+1$ differentiated with respect to a_{t+1}.

Strict concavity. To extend to the case of strict concavity, one needs to exclude the cases in which A_{t+1} and U_{t+1} are linearly dependent. This makes the Cauchy-Schwarz inequality is a strict inequality. The only case in which A_{t+1} and U_{t+1} are linearly dependent is still when $u'''(.)u'(.)/(-u''(.))^2$ is constant and equal to one (Constant Absolute Risk Aversion). In that case, $u'''(.)$ and $(-u''(c_t)^2/u'''(c_t)$ are proportional, and the response of precautionary saving to a change in wealth is zero so $\frac{\partial c_{t+1}}{\partial a_t}$ is a constant.

3.3 Discussion: non-homothetic utility and its interaction with the role of patience and high individual returns

The importance of extending the utility function beyond the case of HARA utility comes from the increased interest in non-homothetic utilities. A number of recent papers show the importance of allowing for non-homotheticities across goods (e.g., basic versus luxury goods à la Wachter and Yogo (2010) or consumption versus bequests à la De Nardi (2004)). Ferrière, Grübener, and Sachs (2025) show that non-homotheticities imply a decreasing relative risk-aversion with respect to total consumption. Meeuwis (2022) suggests implicitly defining a utility function by the relative risk-aversion function $RA(c)$ it implies. As the paper above, he also finds that a non-homothetic utility fits the data best.

A simplified version of his proposed function is

$$RA(c) = \frac{-u''(c)c}{u'(c)} = \gamma_0\big(\frac{c}{\kappa_\beta}\big)^{-\gamma_1}.$$

Such a utility verifies the concavity conditions in Propositions 1 and 2 for the particularly patient consumers or those with particularly high returns such that $R \geq 1$. Indeed, it implies a strictly increasing ratio of prudence over risk-aversion and a strictly decreasing ratio of temperance over prudence.[6] Thus, when $R \geq 1$, Propositions 1 and 2 prove that, under this particular utility function, consumption must be concave in wealth and permanent income. When $R < 1$ consumption may still be concave in both, but it is not necessarily so: the sign of the second-order derivative becomes ambiguous.

Now having $R \geq 1$ means being patient or earning high returns on assets. These are

[6]To see this, note that prudence over risk aversion is $\frac{P(c)}{RA(c)} = \frac{1+\gamma_1+RA(c)}{RA(c)}$. This is strictly increasing in c when $RA(c)$ is strictly decreasing in c. The temperance over prudence is $\frac{T(c)}{P(c)} = \frac{2+\gamma_1+3RA(c)}{1+\gamma_1+RA(c)}$. This is strictly decreasing in c when $RA(c)$ is strictly decreasing in c.

characteristics of wealthier and higher income individuals. The concavity results therefore suggests that non-homotheticicities does have the power to generate concavities, and possibly to amplify them, but more so for those who are more patient and earn higher returns.

4 Precautionary saving and the concavities in wealth and permanent income

4.1 Precautionary saving and wealth

I now examine the reason for the concavity in wealth. When absolute prudence is strictly decreasing, which is a feature of most of the utility functions used in the literature, under assumptions that also ensure concavity, then precautionary saving is strictly decreasing in wealth. This means that, when absolute prudence is strictly decreasing, consumption increases less with wealth at higher levels of wealth because because precautionary decreases less with wealth at higher levels of wealth.

If this is the most common case, it is not the only one. A HARA utility function with strictly increasing absolute prudence makes consumption concave in wealth for the opposite reason, that is, because precautionary saving increases more with wealth at higher levels of wealth.

Lemma 3. I define $\tilde{c}_t = (u')^{-1}(u'(c_t)R^{-1})$. In the model described by (2.1)-(2.6), when absolute prudence is strictly decreasing, then at any $t \leq T-1$

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} > \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

When absolute prudence is constant, at any $t \leq T-1$

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} = \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

When absolute prudence is strictly increasing, then

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} < \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

Proof of Lemma 3. The proofs of Lemma 3 is in the Online Appendix (A.3).

Lemma 4. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, then

$$\frac{-u''(\tilde{c}_t)/u'(\tilde{c}_t)}{-u''(c_t)/u'(c_t)} \geq \frac{-u''(\tilde{c}_t^{PF_t})/u'(\tilde{c}_t^{PF_t})}{-u''(c_t^{PF_t})/u'(c_t^{PF_t})}.$$

Proof of Lemma 4. The proof of Lemma 4 is in the Online Appendix (A.4).

Proposition 3: Precautionary saving and wealth. In the model described by (2.1)-(2.6), when (i) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, and (ii) absolute prudence is strictly decreasing, then precautionary saving decreases strictly with wealth

$$\frac{\partial PS_t}{\partial a_t} < 0.$$

When (iii) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant OR $R = \beta(1+r) = 1$, and (iv) absolute prudence is constant across wealth levels

$$\frac{\partial PS_t}{\partial a_t} = 0.$$

When (v) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \geq 1$ OR this ratio is increasing and $R = \beta(1+r) \leq 1$, and (vi) absolute prudence is strictly increasing, then precautionary saving increases strictly with wealth

$$\frac{\partial PS_t}{\partial a_t} > 0.$$

Proof of Proposition 3 (strictly decreasing absolute prudence). I prove Proposition 3 by backward induction. At $t = T$, $c_t = (1+r)a_t + y_t = c_t^{PF_t}$ so $PS_T = 0$. As a result, $(\partial PS_T)/(\partial a_T) = 0$. Thus, the non-strict version of Proposition 3 is true at T. In that case, Proposition 3 states that $(\partial PS_t)/(\partial a_t) < 0$. This means that the gap between consumption and consumption under perfect foresight decreases with a_t: $(\partial c_t)/(\partial a_t) > (\partial c_t^{PF_t})/(\partial a_t)$. I assume that the non-strict version of Proposition 3 is true at $t+1$ and show that Proposition 3—in its strict version—must then be true at t. I differentiate the Euler equation $u'(c_t) = E_t[u'(c_{t+1})]R$ with respect to a_t. I rearrange using the expression of a_{t+1}. Because the non-strict version of Proposition 3 is true at $t+1$, I have $\frac{\partial c_{t+1}}{\partial a_{t+1}} \geq \frac{\partial c_{t+1}^{PF_{t+1}}}{\partial a_{t+1}} = \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}}$.[7] I obtain

$$\frac{\partial c_t}{\partial a_t} = \frac{\partial a_{t+1}}{\partial a_t} E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)}\right] \tag{4.1}$$

$$\frac{\partial c_t}{\partial a_t} \geq \frac{\partial a_{t+1}}{\partial a_t} E_t\left[\frac{-u''(c_{t+1})}{-u''(\tilde{c}_t)}\right] \tag{4.2}$$

$$\frac{\partial c_t}{\partial a_t} \geq \left((1+r) - \frac{\partial c_t}{\partial a_t}\right) \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} E_t\left[\frac{-u''(c_{t+1})}{-u''(c_t)}\right] \tag{4.3}$$

$$\frac{\partial c_t}{\partial a_t} \geq (1+r) \frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} E_t\left[\frac{-u''(c_{t+1})}{-u''(c_t)}\right] R}{1 + \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} E_t\left[\frac{-u''(c_{t+1})}{-u''(c_t)}\right] R}. \tag{4.4}$$

When absolute prudence is strictly decreasing, and because the function $f(x) = x/(1+x)$ is increasing, Lemma 3 implies that

$$\frac{\partial c_t}{\partial a_t} \geq (1+r) \frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}}{1 + \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}}. \tag{4.5}$$

Lemma 4 then implies

$$\frac{\partial c_t}{\partial a_t} \geq (1+r) \frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t^{PF_t}))/u'(\tilde{c}_t^{PF_t})}{(-u''(c_t^{PF_t}))/u'(c_t^{PF_t})}}{1 + \frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t^{PF_t}))/u'(\tilde{c}_t^{PF_t})}{(-u''(c_t^{PF_t}))/u'(c_t^{PF_t})}} = \frac{\partial c_t^{PF_t}}{\partial a_t}. \tag{4.6}$$

Proof of Proposition 3 (constant absolute prudence). At $t = T$, $c_t = (1+r)a_t + y_t = c_t^{PF_t}$ so $PS_T = 0$. As a result, $(\partial PS_T)/(\partial a_T) = 0$. I assume that precautionary saving at $t+1$ does not change with wealth at $t+1$, so and show that precautionary saving at t must then

[7] Indeed, because under perfect foresight consumption is linear in wealth with the linearity coefficient independent of past wealth and income, the marginal propensity to consume is the same under perfect foresight at t and under perfect foresight at $t+1$.

not change with wealth at t. With the same reasoning as above

$$\frac{\partial c_t}{\partial a_t} = (1+r)\frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} E_t\left[\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R}{1+\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} E_t\left[\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R}. \tag{4.7}$$

When absolute prudence is constant, Lemma 3 implies

$$\frac{\partial c_t}{\partial a_t} = (1+r)\frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}}{1+\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}}. \tag{4.8}$$

When absolute prudence is constant and $u'''(.)u'(.)/(-u''(.))$ is constant, then absolute risk aversion $-u''(.)/u'(.)$ is constant so it takes the same value in the presence of uncertainty and under perfect foresight. Else if $R = 1$, then by definition $\tilde{c} = c$ and the ratio $\frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}$ equals one both in the presence of uncertainty and under perfect foresight. In both cases

$$\frac{\partial c_t}{\partial a_t} = (1+r)\frac{\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t^{PF_t}))/u'(\tilde{c}_t^{PF_t})}{(-u''(c_t^{PF_t}))/u'(c_t^{PF_t})}}{1+\frac{\partial c_{t+1}^{PF_t}}{\partial a_{t+1}} \frac{(-u''(\tilde{c}_t^{PF_t}))/u'(\tilde{c}_t^{PF_t})}{(-u''(c_t^{PF_t}))/u'(c_t^{PF_t})}} = \frac{\partial c_t^{PF_t}}{\partial a_t}. \tag{4.9}$$

Proof of Proposition 3 (strictly increasing absolute prudence). The same reasoning as in the case of decreasing absolute prudence yields the results.

4.2 Precautionary saving and the concavity in permanent earnings

Lemma 5. In the model described by (2.1)-(2.6), when relative risk-aversion is constant OR relative risk-aversion is decreasing and $R \leq 1$ OR relative risk-aversion is increasing and $R \geq 1$, and relative prudence is increasing, then at any $t \leq T-1$

$$E_t[c_{t+1}(-u''(c_{t+1}))]R \leq c_t(-u''(c_t)).$$

Proof of Lemma 5. The proof of Lemma 5 is in the Online Appendix (A.5).

Proposition 5: Super-homogeneity. In the model described by (2.1)-(2.6), when relative risk-aversion is constant OR relative risk-aversion is decreasing and $R \geq 1$ OR relative risk-aversion is increasing and $R \leq 1$, and relative prudence is increasing, then at any $t \leq T$ consumption is sub-homogeneous of degree one in wealth and permanent income, that is, smaller than the weighted sum of its derivatives with respect to wealth and permanent income

$$a_t \frac{\partial c_t}{\partial a_t} + e^{p_t} \frac{\partial c_t}{\partial e^{p_t}} \leq c_t.$$

Proof of Proposition 5. The proof of Proposition 5 is in the Online Appendix (A.6).

Proposition 6: Precautionary saving and permanent income. In the model described by (2.1)-(2.6), when wealth is strictly positive, precautionary saving decreases with wealth and less so at a higher level of wealth

$$\frac{\partial PS_t}{\partial a_t} < 0 \text{ and } \frac{\partial^2 PS_t}{\partial a_t^2} > 0.$$

Proof of Proposition 6. From Proposition 5, consumption is sub-homogeneous of degree one in wealth and permanent earnings. This means that

$$\frac{\partial c_t}{\partial e^{p_t}} \leq \frac{c_t}{e^{p_t}} - \frac{a_t}{e^{p_t}} \frac{\partial c_t}{\partial a_t} \leq \frac{c_t^{PF_t}}{e^{p_t}} - \frac{a_t}{e^{p_t}} \frac{\partial c_t^{PF_t}}{\partial a_t} = \frac{\partial c_t^{PF_t}}{\partial e^{p_t}}. \tag{4.10}$$

The last equality is because consumption is homogeneous of degree one under perfect foresight at t

$$c_t^{PF_t} = a_t \frac{\partial c_t^{PF_t}}{\partial a_t} + e^{p_t} \frac{\partial c_t^{PF_t}}{\partial e^{p_t}}. \tag{4.11}$$

5 Conclusion

In this paper, I establish analytically that it is possible for precautionary saving to increase with a certain type of resources, namely with the permanent component of income. The reason is that an increase in permanent income raises the worker's exposure to future shocks and increases its desire to save.

Although this wealth accumulation is eventually self-defeating, since more wealth then reduce the desire to save, for finite-lived consumers it can be long before the saving rates of the high permanent income consumers get as low as those of the low permanent income consumers.

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A Online Appendix

A.1 Lemma 1

Lemma 1. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when (i) the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, and (ii) the ratio of temperance over prudence $(-u''''(.))(-u''(.))/u'''(.)^2$ is constant or decreasing

$$E_t[(-u''(c_{t+1}))^2/u'''(c_{t+1})]R \leq (-u''(c_t))^2/u'''(c_t).$$

Proof of Lemma 1. I denote φ_t^f the equivalent premium at t of a utility function $f(.)$ associated with uncertainty about consumption c_{t+1}, that is, the value such that $E_t[f(c_{t+1})] = u(E[c_{t+1}] - \varphi^f)$. I call this value the 'premium associated with $f(.)$' in the remainder of the proof.

I define $\tilde{c}_t$ such that $u'(\tilde{c}_t) = u'(c_t)R^{-1}$. The Euler equation of the model implies

$$u'(c_t) = E_t[u'(c_{t+1})]R = u'(E_t[c_{t+1}] - \varphi_t^{u'})R \tag{A.1}$$
$$u'(c_t)R^{-1} = u'(E_t[c_{t+1}] - \varphi_t^{u'}) \tag{A.2}$$
$$u'(\tilde{c}_t) = u'(E_t[c_{t+1}] - \varphi_t^{u'}) \tag{A.3}$$
$$\tilde{c}_t = E_t[c_{t+1}] - \varphi_t^{u'}. \tag{A.4}$$

The definition of $\tilde{c}_t$ also implies $R = \frac{u'(c_t)}{u'(\tilde{c}_t)}$. I denote $g(.) = (-u''(.))^2/u'''(.)$.

When the ratio of temperance over prudence $(-u''''(.))(-u''(.))/(u'''(.))^2$ is constant, then $g'(.)/u''(.) = -((-u''''(.))(-u''(.))/u'''(.)^2 - 2)$ is constant. Pratt (1964) then implies that $\varphi_t^{u'} = \varphi_t^g$. As a result

$$E_t[(-u''(c_{t+1}))^2/u'''(c_{t+1})] = E_t[g(c_{t+1})] = g(E_t[c_{t+1}] - \varphi_t^g) \tag{A.5}$$
$$= g(E_t[c_{t+1}] - \varphi_t^{u'}) = g(\tilde{c}_t) = (-u''(\tilde{c}_t))^2/u'''(\tilde{c}_t).$$

When the ratio of temperance over prudence $(-u''''(.))(-u''(.))/(u'''(.))^2$ is decreasing, then $g'(.)/u''(.) = -((-u''''(.))(-u''(.))/u'''(.)^2 - 2)$ is decreasing. I first assume that this ratio is also smaller than two over all possible values of c_{t+1} and explain below how other cases solve in a similar manner. This means that $g'(.)/u''(.)$ is positive. From Lemma 1 in Commault (2025), because $g'(.)/u''(.)$ is positive and decreasing the premium associated with $g(.)$ is lower than the premium associated with $u'(.)$: $\varphi_t^g \leq \varphi_t^{u'}$. Also, because $g'(.)/u''(.)$ is strictly positive, then $g'(.)$ is strictly negative and $g(.)$ is strictly decreasing. As a result

$$E_t[(-u''(c_{t+1}))^2/u'''(c_{t+1})] = E_t[g(c_{t+1})] = g(E_t[c_{t+1}] - \varphi_t^g) \tag{A.6}$$
$$\leq g(E_t[c_{t+1}] - \varphi_t^{u'}) = g(\tilde{c}_t) = (-u''(\tilde{c}_t))^2/u'''(\tilde{c}_t).$$

If the ratio is larger than two over all possible values of c_{t+1}, then $\varphi_t^g \geq \varphi_t^{u'}$ but $g(.)$ is increasing over that domain so one obtains the same inequality. When the ratio is neither always above or always below two, one can define the premiums over the domains where the ratio is. Writing the expected value of $g(.)$ as the sum of its expected value over the two domains, one obtains the same expression.

Now, to obtain the final inequality, I divide both side by $(-u''(c_t))^2/u'''(c_t)$—which is strictly non-zero from the model's assumptions—, multiply both sides by R, and substitute for $R = \frac{u'(c_t)}{u'(\tilde{c}_t)}$

$$\frac{E_t[(-u''(c_{t+1}))^2/u'''(c_{t+1})]}{(-u''(c_t))^2/u'''(c_t)}R \leq \frac{(-u''(\tilde{c}_t))^2/u'''(\tilde{c}_t)}{(-u''(c_t))^2/u'''(c_t)}R \tag{A.7}$$

$$\leq \frac{(-u''(\tilde{c}_t))^2/u'''(\tilde{c}_t)}{(-u''(c_t))^2/u'''(c_t)} \frac{u'(c_t)}{u'(\tilde{c}_t)} \tag{A.8}$$

$$\leq \frac{u'''(c_t)u'(c_t)/(-u''(c_t))^2}{u'''(\tilde{c}_t)u'(\tilde{c}_t)/(-u''(\tilde{c}_t))^2} \leq 1. \tag{A.9}$$

Indeed, when the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant, then its value in c_t is the same as its value in $\tilde{c}_t$. The final ratio is exactly equal to one. When the ratio of prudence over risk-aversion is decreasing, and when $R \leq 1$ so $\tilde{c}_t \leq c_t$, then prudence over risk-aversion is smaller in c_t than in $\tilde{c}_t$ and the final ratio is smaller than one. When the ratio of prudence over risk-aversion is increasing, and when $R \geq 1$ so $c_t \leq \tilde{c}_t$, then prudence over risk-aversion is also smaller in c_t than in $\tilde{c}_t$ and the final ratio is also smaller than one.

A.2 Lemma 2

Lemma 2. In the model described above by (2.1)-(2.6) with $R = \beta(1+r) \leq 1$, at any period $t < T$

$$\frac{\partial^2 c_t}{\partial a_t \partial e^{p_t}} \leq \sqrt{\frac{\partial^2 c_t}{\partial a_t^2} \frac{\partial^2 c_t}{\partial (e^{p_t})^2}}.$$

Proof of Lemma 2. I prove Lemma 2 by backward induction. At the last period $t = T$, $\frac{\partial^2 c_T}{\partial a_T^2} = \frac{\partial^2 c_T}{\partial e^{p_T 2}} = \frac{\partial^2 c_T}{\partial a_T \partial e^{p_T}} = 0$ so Lemma 2 holds true not strictly. I assume it holds true not strictly at $t+1$ and show it must then hold true strictly at t.

Expressing the second-order partial derivatives as a sum of two terms (Y and X). I derive each side of the Euler equation with respect to a_t then with respect to e^{p_t}.

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial a_t \partial e^{p_t}}(-u''(c_t)) - \frac{\partial c_t}{\partial a_t}\frac{\partial c_t}{\partial e^{p_t}}u'''(c_t) = \\
&\quad E_t\left[\left(\frac{\partial^2 a_{t+1}}{\partial a_t \partial e^{p_t}}\frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{\partial a_{t+1}}{\partial a_t}\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + \frac{\partial a_{t+1}}{\partial a_t}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\right)(-u''(c_{t+1}))\right] \\
&\quad - E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{\partial c_{t+1}}{\partial e^{p_t}}u'''(c_{t+1})\right].
\end{aligned} \tag{A.10}$$

Using the differentiated budget constraint, I substitute $\frac{\partial^2 a_{t+1}}{\partial a_t \partial e^{p_t}} = -\frac{\partial^2 c_t}{\partial a_t \partial e^{p_t}}$. Using the differentiated Euler equation with respect to a_t and with respect to e^{p_t}, I substitute $\partial c_t/\partial a_t = E_t[(\partial c_{t+1}/\partial a_t)(-u''(c_{t+1}))/(-u''(c_t))]$ and $(\partial c_t/\partial e^{p_t}) = E_t[(\partial c_{t+1}/\partial e^{p_t})(-u''(c_{t+1}))/(-u''(c_t))]$. I rearrange

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial a_t \partial e^{p_t}}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) \\
&\quad = E_t\left[\left(\frac{\partial a_{t+1}}{\partial a_t}\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + \frac{\partial a_{t+1}}{\partial a_t}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\right)\frac{-u''(c_{t+1})}{-u''(c_t)}\right] \\
&\quad - \frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right)
\end{aligned} \tag{A.11}$$

I derive each side of the Euler equation twice with respect to a_t. With a similar rearranging

$$\begin{aligned}
\frac{d^2 c_t}{da_t^2}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) &= E_t\left[(\frac{da_{t+1}}{da_t})^2\frac{d^2 c_{t+1}}{da_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\right] \\
&\quad - \frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[(\frac{dc_{t+1}}{da_t})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{dc_{t+1}}{da_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\right)
\end{aligned} \tag{A.12}$$

I derive each side of the Euler equation twice with respect to e^{p_t}. With a similar rearranging

$$\begin{aligned}
&\frac{\partial^2 c_t}{\partial (e^{p_t})^2}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) \\
&\quad = E_t\left[\left((\frac{\partial a_{t+1}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + 2\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + (\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial {e^{p_{t+1}}}^2}\right)\frac{-u''(c_{t+1})}{-u''(c_t)}\right] \\
&\quad - \frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[(\frac{\partial c_{t+1}}{\partial e^{p_t}})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\right)
\end{aligned} \tag{A.13}$$

I define the following notations

$$Y_t^{ap} = E_t\left[\left(\frac{\partial a_{t+1}}{\partial a_t}\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + \frac{\partial a_{t+1}}{\partial a_t}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\right)\frac{-u''(c_{t+1})}{-u''(c_t)}\right];$$

$$Y_t^{a} = E_t\left[(\frac{\partial a_{t+1}}{\partial a_t})^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}\right];$$

$$Y_t^{p} = E_t\left[\left((\frac{\partial a_{t+1}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2} + 2\frac{\partial a_{t+1}}{\partial e^{p_t}}\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} + (\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\right)\frac{-u''(c_{t+1})}{-u''(c_t)}\right].$$

and

$$X_t^{ap} = -\frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right);$$

$$X_t^{a} = -\frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[(\frac{\partial c_{t+1}}{\partial a_t})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{\partial c_{t+1}}{\partial a_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\right);$$

$$X_t^{p} = -\frac{u'''(c_t)}{-u''(c_t)}\left(E_t\left[(\frac{\partial c_{t+1}}{\partial e^{p_t}})^2\frac{u'''(c_{t+1})}{u'''(c_t)}\right] - E_t\left[\frac{\partial c_{t+1}}{\partial e^{p_t}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]^2\right).$$

I thus have

$$\frac{\partial^2 c_t}{\partial a_t \partial e^{p_t}}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) = Y_t^{ap} + X_t^{ap} \tag{A.14}$$

$$\frac{\partial^2 c_t}{\partial a_t^2}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) = Y_t^{a} + X_t^{a} \tag{A.15}$$

$$\frac{\partial^2 c_t}{\partial e^{p_t 2}}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right) = Y_t^{p} + X_t^{p}. \tag{A.16}$$

Proving the inequality of the Y terms. I prove that $(Y_t^{ap})^2 < Y_t^a Y_t^p$. I compute $Y_t^a Y_t^p$

$$Y_t^a Y_t^p = (\frac{\partial a_{t+1}}{\partial a_t})^2(\frac{\partial a_{t+1}}{\partial e^{p_t}})^2 E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}] \tag{A.17}$$

$$+ 2(\frac{\partial a_{t+1}}{\partial a_t})^2\frac{\partial a_{t+1}}{\partial e^{p_t}}E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\frac{-u''(c_{t+1})}{-u''(c_t)}]$$

$$+ (\frac{\partial a_{t+1}}{\partial a_t})^2 E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[(\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}})^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\frac{-u''(c_{t+1})}{-u''(c_t)}].$$

I also compute $(Y_t^{ap})^2$

$$(Y_t^{ap})^2 = \big(\frac{\partial a_{t+1}}{\partial a_t}\big)^2\big(\frac{\partial a_{t+1}}{\partial e^{p_t}}\big)^2 E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}] \\ +2\big(\frac{\partial a_{t+1}}{\partial a_t}\big)^2\frac{\partial a_{t+1}}{\partial e^{p_t}}E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\frac{-u''(c_{t+1})}{-u''(c_t)}] \\ +\big(\frac{\partial a_{t+1}}{\partial a_t}\big)^2 E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2. \tag{A.18}$$

Because Lemma 2 is true, not necessarily strictly, at $t+1$, I have $\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}} \leq \sqrt{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}$

$$E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2 \leq E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\sqrt{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2 \tag{A.19}$$

By Cauchy-Schwarz, $E[AB]^2 \leq E[A^2]E[B^2]$. I denote $A = \sqrt{\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}}\sqrt{\frac{-u''(c_{t+1})}{-u''(c_t)}}$ and $B = \frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\sqrt{\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}}\sqrt{\frac{-u''(c_{t+1})}{-u''(c_t)}}$. Both are strictly uncertain because c_{t+1} is. I thus have

$$E_t[\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\frac{\partial^2 c_{t+1}}{\partial a_{t+1}\partial e^{p_{t+1}}}\frac{-u''(c_{t+1})}{-u''(c_t)}]^2 \leq E_t[\frac{\partial^2 c_{t+1}}{\partial a_{t+1}^2}\frac{-u''(c_{t+1})}{-u''(c_t)}]E_t[\big(\frac{\partial e^{p_{t+1}}}{\partial e^{p_t}}\big)^2\frac{\partial^2 c_{t+1}}{\partial (e^{p_{t+1}})^2}\frac{-u''(c_{t+1})}{-u''(c_t)}] \tag{A.20}$$

This means that, while the first two terms in expressions (A.17) and (A.18) are the same, the third one is strictly smaller in the expression of $(Y_t^{ap})^2$ than in the expression of $Y_t^a Y_t^p$. As a result

$$0 \leq (Y_t^{ap})^2 \leq Y_t^a Y_t^p. \tag{A.21}$$

Proving the inequality of the X terms. I then show that $(X_t^{ap})^2 < X_t^a X_t^p$. I denote $U_{t+1} \equiv \frac{u'''(c_{t+1})/(-u''(c_{t+1}))^2}{u'''(c_t)/(-u''(c_t))^2}$ and $< X,Y > \equiv E_t[XYU_{t+1}] - E_t[X]E_t[Y]$. From Lemma 1, because the ratio of temperance over prudence and prudence over risk-aversion are constant or decreasing, I have $E_t[U_{t+1}^{-1}] \leq 1$. As a result, the relation $< .,. >$ verifies $< X,X > = E_t[X^2U_{t+1}] - E_t[X]^2 \geq E_t[X^2U_{t+1}]E_t[U_{t+1}^{-1}] - E_t[X]^2 > 0$ for any $X \neq 0$. The relation $< .,. >$ also verifies symmetry and linearity. It therefore defines an inner product space. As a result, the Cauchy-Schwarz inequality applies to this relation

$$
\begin{aligned}
&\Big|\langle \frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)}, \frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} \frac{-u''(c_{t+1})}{-u''(c_t)} \rangle\Big|^2 \\
&\leq \langle \frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)}, \frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)} \rangle \times \langle \frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} \frac{-u''(c_{t+1})}{-u''(c_t)}, \frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} \frac{-u''(c_{t+1})}{-u''(c_t)} \rangle.
\end{aligned} \tag{A.22}
$$

Multiplying both sides by $(-\frac{u'''(c_t)}{-u''(c_t)})^2$ I have

$$
0 \leq (X_t^{ap})^2 \leq X_t^a X_t^p. \tag{A.23}
$$

Combining the inequalities of the two sets of terms. The first result implies $Y_t^a Y_t^p > (Y_t^{ap})^2$ and $|Y_t^{ap}| < \sqrt{Y_t^a Y_t^b}$—where $\sqrt{Y_t^a Y_t^b}$ is defined because $Y_t^a Y_t^b \geq 0$. The second result implies $X_t^a X_t^p > (X_t^{ap})^2$ and $|X_t^{ap}| < \sqrt{X_t^a X_t^b}$—where $\sqrt{X_t^a X_t^b}$ is defined because $X_t^a X_t^b \geq 0$. Lemma 2 holds true strictly at t

$$
\begin{aligned}
&\left(\frac{\partial^2 c_t}{\partial a_t^2} \frac{\partial^2 c_t}{\partial (e^{p_t})^2} - (\frac{d^2 c_t}{\partial a_t \partial e^{p_t}})^2\right)\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{-u''(c_{t+1})}{-u''(c_t)}\right]\right)^2 \\
&= \underbrace{Y_t^a Y_t^p}_{\geq (Y_t^{ap})^2} + Y_t^a X_t^p + X_t^a Y_t^p + \underbrace{X_t^a X_t^p}_{\geq (X_t^{ap})^2} - \left((Y_t^{ap})^2 + 2 \underbrace{Y_t^{ap} X_t^{ap}}_{\leq \sqrt{Y_t^a Y_t^b}\sqrt{X_t^a X_t^p}} + (X_t^{ap})^2\right) \\
&\geq Y_t^a X_t^p + X_t^a Y_t^p - 2\sqrt{Y_t^a Y_t^p X_t^a X_t^p} \\
&\geq \left(\sqrt{Y_t^a X_t^p} - \sqrt{X_t^a Y_t^p}\right)^2 \geq 0
\end{aligned} \tag{A.24}
$$

The result that $Y_t^{ap} X_t^{ap} \leq \sqrt{Y_t^a Y_t^b}\sqrt{X_t^a X_t^p}$ holds true because if $Y_t^{ap} = |Y_t^{ap}|$ and $X_t^{ap} = |X_t^{ap}|$, it directly stems from previous results. If only one of the terms Y_t^{ap} and X_t^{ap} is negative, then their product is negative and also smaller than $\sqrt{Y_t^a Y_t^b}$. If both are negative, then their product is equal to the product of their absolute values and previous results also directly imply this result. Thus Lemma 2 holds true at any period $t \leq T$

A.3 Lemma 3

Lemma 3. I define $\tilde{c}_t = (u')^{-1}(u'(c_t)R^{-1})$. In the model described by (2.1)-(2.6), when absolute prudence is strictly decreasing, then at any $t \leq T-1$

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} > \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

When absolute prudence is constant, at any $t \leq T-1$

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} = \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

When absolute prudence is strictly increasing, then

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} < \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)}.$$

Proof of Lemma 3. As in the proof of Lemma 1, I define $\tilde{c}_t$ such that $u'(c_t)R^{-1} \equiv u'(\tilde{c}_t)$. With $R \leq 1$, this means $c_t \geq \tilde{c}_t$. The Euler equation of the model implies

$$u'(c_t) = E_t[u'(c_{t+1})]R \tag{A.25}$$

$$u'(\tilde{c}_t) = E_t[u'(c_{t+1})] \tag{A.26}$$

$$\tilde{c}_t = E_t[c_{t+1}] - \varphi_t^{u'}. \tag{A.27}$$

The definition of $\tilde{c}_t$ also implies $R = \frac{u'(c_t)}{u'(\tilde{c}_t)}$. I consider $h(.) = -u''(.)$ and $l(.) = u'(.)$. Their ratio is $h'(.)/l'(.) = u'''(.)/-u''(.)$, which is the level of absolute prudence.

In the case where absolute risk aversion is constant, utility displays CARA. Absolute prudence is then constant as well. The ratio $h'(.)/l'(.)$ is constant, so $\varphi^{-u''} = \varphi^{u'}$. As a result

$$E_t[-u''(c_{t+1})] = -u''(E_t[c_{t+1}] - \varphi^{-u''}) = -u''(E_t[c_{t+1}] - \varphi^{u'}) = -u''(\tilde{c}_t). \tag{A.28}$$

I multiply both sides by $R = (u'(c_t)/u'(\tilde{c}_t))$ and divide both side by $(-u''(c_t))$.

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} = \frac{-u''(\tilde{c}_t)}{-u''(c_t)}R \tag{A.29}$$

$$= \frac{-u''(\tilde{c}_t)}{-u''(c_t)}\frac{u'(c_t)}{u'(\tilde{c}_t)} \qquad = \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)} = 1. \tag{A.30}$$

Indeed, when absolute risk-aversion $(-u''(.)/u'(.)$ is constant, its value is the same in $c_t \geq \tilde{c}_t$ than in $\tilde{c}_t$.

In the case where absolute prudence is strictly decreasing, because by assumption is also strictly positive, the Lemma 1 in **Commault2025** implies that $\varphi^{-u''} > \varphi^{u'}$. Because $u''' > 0$, $-u''(.)$ is decreasing. As a result

$$E_t[-u''(c_{t+1})] = -u''(E_t[c_{t+1}] - \varphi^{-u''}) > -u''(E_t[c_{t+1}] - \varphi^{u'}) = -u''(\tilde{c}_t). \tag{A.31}$$

I multiply both sides by $R = (u'(c_t)/u'(\tilde{c}_t))$ and divide both side by $(-u''(c_t))$.

$$\frac{E_t[-u''(c_{t+1})]R}{-u''(c_t)} > \frac{-u''(\tilde{c}_t)}{-u''(c_t)}R \tag{A.32}$$

$$> \frac{-u''(\tilde{c}_t)}{-u''(c_t)}\frac{u'(c_t)}{u'(\tilde{c}_t)} \qquad > \frac{(-u''(\tilde{c}_t))/u'(\tilde{c}_t)}{(-u''(c_t))/u'(c_t)} \geq 1. \tag{A.33}$$

When absolute risk-aversion $(-u''(.)/u'(.)$ is strictly decreasing, its value is smaller in $c_t \geq \tilde{c}_t$ than in $\tilde{c}_t$.

A.4 Lemma 4

Lemma 4. In the model described above by (2.1)-(2.6), at any period $t \leq T$, when the ratio of prudence over risk-aversion $u'''(.)u'(.)/(-u''(.))^2$ is constant (HARA utility) OR this ratio is decreasing and $R = \beta(1+r) \leq 1$ OR this ratio is increasing and $R = \beta(1+r) \geq 1$, then

$$\frac{-u''(\tilde{c}_t)/u'(\tilde{c}_t)}{-u''(c_t)/u'(c_t)} \geq \frac{-u''(\tilde{c}_t^{PF_t})/u'(\tilde{c}_t^{PF_t})}{-u''(c_t^{PF_t})/u'(c_t^{PF_t})}.$$

Proof of Lemma 4. Because $\tilde{c}_t$ is entirely defined by c_t when the parameter R is give, the ratio $\frac{-u''(\tilde{c}_t)/u'(\tilde{c}_t)}{-u''(c_t)/u'(c_t)}$ is also entirely a function of c_t. I denote $h(c_t)$ this function. When utility is HARA, this ratio is constant. When prudence over risk-aversion is decreasing and

$R=\beta(1+r)\leq 1$, this function is decreasing so it is higher in the presence of uncertainty $c_t < c_t^{PF_t}$. When prudence over risk-aversion is increasing and $R=\beta(1+r)\geq 1$, this function is increasing so it is higher in the presence of uncertainty $c_t < c_t^{PF_t}$.

A.5 Lemma 5

Lemma 5. In the model described by (2.1)-(2.6), when relative risk-aversion is constant OR relative risk-aversion is decreasing and $R\leq 1$ OR relative risk-aversion is increasing and $R\geq 1$, and relative prudence is increasing, then at any $t\leq T-1$

$$E_t[c_{t+1}(-u''(c_{t+1}))]R \leq c_t(-u''(c_t)).$$

Proof of Lemma 5. The proof of Lemma 5 builds on a similar reasoning as the proof of Lemma 1, with a different function.

A.6 Proposition 5

Proposition 5: Super-homogeneity. In the model described by (2.1)-(2.6), when relative risk-aversion is constant OR relative risk-aversion is decreasing and $R\geq 1$ OR relative risk-aversion is increasing and $R\leq 1$, and relative prudence is increasing, then at any $t\leq T$ consumption is sub-homogeneous of degree one in wealth and permanent income, that is, smaller than the weighted sum of its derivatives with respect to wealth and permanent income

$$a_t\frac{\partial c_t}{\partial a_t}+e^{p_t}\frac{\partial c_t}{\partial e^{p_t}}\leq c_t.$$

Proof of Proposition 5. I prove Proposition 5 by backward induction. At the last period $t=T$ people consume everything they have so $c_T=(1+r)a_T+e^{\varepsilon_T}e^{p_T}=(\partial c_T/\partial a_T)a_T+(\partial c_T/\partial e^{p_T})e^{p_T}$. This means that Proposition 5 holds true at $t=T$. I assume that it holds true at t+1, and show that it must then hold true at t. I differentiate both sides of the Euler equation with respect to e^{p_t}. I rearrange the expression using that $e^{\varepsilon_t}=(a_{t+1}-(1+r)a_t+$

$c_t)/e^{p_t}$ from the budget constraint

$$\frac{\partial c_t}{\partial e^{p_t}} = E_t\big[\big((-\frac{\partial c_t}{\partial e^{p_t}} + e^{\varepsilon_t})\frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{e^{p_{t+1}}}{e^{p_t}}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}}\big)\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R \tag{A.34}$$

$$\frac{\partial c_t}{\partial e^{p_t}}\left(1 + E_t\big[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R\right)$$

$$= E_t\big[\big(e^{\varepsilon_t}\frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{e^{p_{t+1}}}{e^{p_t}}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}}\big)\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R \tag{A.35}$$

$$= E_t\big[\big(\frac{a_{t+1} - (1+r)a_t + c_t}{e^{p_t}}\frac{\partial c_{t+1}}{\partial a_{t+1}} + \frac{e^{p_{t+1}}}{e^{p_t}}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}}\big)\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R \tag{A.36}$$

$$= \frac{1}{e^{p_t}}E_t\left[\left(a_{t+1}\frac{\partial c_{t+1}}{\partial a_{t+1}} + e^{p_{t+1}}\frac{\partial c_{t+1}}{\partial e^{p_{t+1}}} + (-(1+r)a_t + c_t)\frac{\partial c_{t+1}}{\partial a_{t+1}}\right)\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R \tag{A.37}$$

I then use that Proposition 5 holds true at t+1 to substitute $(\partial c_{t+1}/\partial a_{t+1})a_{t+1} + (\partial c_{t+1}/\partial e^{p_{t+1}})e^{p_{t+1}}$ as lower than c_{t+1}. Finally, I use Lemma 5 to substitute $E_t\big[(c_{t+1}(-u''(c_{t+1}))/(c_t(-u''(c_t))\big]R \leq 1$.

$$\frac{\partial c_t}{\partial e^{p_t}}\left(1 + E_t\big[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R\right)$$

$$\leq \frac{c_t}{e^{p_t}}E_t\left[\frac{c_{t+1}}{c_t}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R + \frac{-(1+r)a_t + c_t}{e^{p_t}}E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R \tag{A.38}$$

$$\leq \frac{c_t}{e^{p_t}} + \frac{-(1+r)a_t + c_t}{e^{p_t}}E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R. \tag{A.39}$$

$$\leq \frac{c_t}{e^{p_t}}\left(1 + E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R\right) - \frac{a_t}{e^{p_t}}(1+r)E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R. \tag{A.40}$$

Then, from differentiating both sides of the Euler equation with respect to a_t, I have

$$(1+r)E_t\left[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\right]R = \frac{\partial c_t}{\partial a_t}\left(1 + E_t\big[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R\right). \tag{A.41}$$

I use (A.41) to substitute in (A.40) and divide all sides by $\left(1 + E_t\big[\frac{\partial c_{t+1}}{\partial a_{t+1}}\frac{-u''(c_{t+1})}{-u''(c_t)}\big]R\right)$. I obtain

$$\frac{\partial c_t}{\partial e^{p_t}} \leq \frac{c_t}{e^{p_t}} - \frac{a_t}{e^{p_t}}\frac{\partial c_t}{\partial a_t}. \tag{A.42}$$

This means that $e^{p_t}\frac{\partial c_t}{\partial e^{p_t}} + a_t\frac{\partial c_t}{\partial a_t} \leq c_t$. Thus, Proposition 5 holds true at t. By backward induction, it must then hold true at any period $t \leq T$.